GEOMETRY OF TWO-COMPONENT SPINORS

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1. The differential equations of Dirac[1] for an electron have been generalized by Weyl[2] and Fock[3] to the case of general relativity and further studied by Schouten[4] and others. The theory which results makes use of ideas which seem likely to be of considerable significance in differential geometry in general. It is the purpose of this note to set forth some of the requisite geometrical ideas in the simplest case, the case which can be handled by direct generalization of the spinors defined by van der Waerden.[5] The note is thus a sort of geometric commentary on the paper of Weyl.

At each point of an *underlying manifold* with a coördinate system (x^1, ..., x^4) there is a *tangent space* with coördinates (dx^1, ..., dx^4). A differential form $g_{ij}dx^i dx^j$, gives a Euclidean measure of distance, angle, etc., in each tangent space. The Riemannian geometry is the simultaneous theory of all these Euclidean spaces.

The locus

$$g_{ij}dx^i dx^j = 0 \tag{1.1}$$

is a quadratic cone through the origin. Let us confine ourselves to the relativity case, in which (1.1) is the *light-cone* and is of signature ($-$, $-$, $-$, $+$). This means that there is a transformation of coördinates, $dx \longrightarrow X$, in the tangent spaces which carries the cone (1.1) into

$$-(X^1)^2 - (X^2)^2 - (X^3)^2 + (X^4)^2 = 0. \tag{1.2}$$

Let us write the equations of this transformation in the form,

$$X^a = g^a_i dx^i. \tag{1.3}$$

We adopt the convention that the letters a, ..., h used as indices refer to the coördinate system X in the tangent space, and i, ..., z used as indices refer to the coördinates of the underlying space or of the differentials dx^1, ..., dx^4. We shall also use as far as conveniently possible, the convention of Schouten that a geometrical object shall be indicated in all coördinate systems by the same carrier letter, different sorts of coördinate systems being indicated by different alphabets or portions of alphabets used as indices. Thus the quadratic form in (1.2) is

$$g_{ab}X^a X^b.$$

Also we have

$$g_{ab}g^a_i g^b_j = g_{ij},$$

where

$$(g_{ab}) = \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1. \end{matrix} \tag{1.4}$$

We make the definition

$$g^{ia} = g^{ai} = g^{ij} g^a_j \tag{1.5}$$

and

$$g^i_a = g_{ab} g^{ib}, \tag{1.6}$$

where, as usual

$$g^{ik} g_{jk} = \delta^i_j.$$

It then follows that

$$g^i_j = g^i_a g^a_j = \delta^i_j \text{ and } g^i_a g^b_i = g^b_a = \delta^a_b. \tag{1.7}$$

Having the existence of the quantities g^a_j in one coördinate system x we are free to assume a transformation law for them. We assume that g^1_i, g^2_i, g^3_i, g^4_i are four covariant vectors. It then follows that the components of these vectors in any other coördinate system will transform the equation (1.1) in the new coördinate system into (1.2). If we assume also that g^i_1, g^i_2, g^i_3, g^i_4 are four contravariant vectors, then (1.5), (1.6) and (1.7) hold in all coördinate systems.

From the fact that g^1_i, g^2_i, g^3_i, g^4_i are covariant vectors, it follows that X^1, X^2, X^3, X^4 are unaltered by any transformation of coördinates x in the underlying space. We therefore call them *scalar coördinates* for the tangent Euclidean space. They can be subjected to an arbitrary Lorentz transformation

$$\overline{X}^a = L^a_b X^b \tag{1.8}$$

without altering anything that has been said, provided that we replace (1.3) by

$$\overline{X}^a = \bar{g}^a_j dx^j$$

where

$$\bar{g}^a_j = L^a_b g^b_j.$$

With respect to transformations of coördinates in the underlying space the quantities L^a_b are 16 scalars.

The quantities g_{ai}, g^a_i, g^i_a, etc., are components of the metric tensor with respect to a in the scalar coördinate system X, and with respect to i in the coördinate system dx.

2. The cone (1.1) is a system of straight lines through its vertex. The structure of this system of lines may be made evident as follows. Consider any 3-space S_3 in the tangent 4-space but not passing through the origin (point of contact). Each line of the cone cuts S_3 in one and but

one point and the set of all these points constitutes a quadric surface in S_3. Hence the relationships among themselves of the lines of the cone are the same as those among the points of an ordinary quadric in a 3-space.

The property of a quadric which we wish to use is that it contains two one-parameter systems of straight lines, the generators of the quadric. Any line of one system cuts all the lines of the other system, and no two lines of the same system intersect. If λ is a parameter which designates a line of the first system and μ a parameter which designates a line of the second system then the pair of numbers (λ, μ) designates a unique point on the quadric. Inversely any point of the quadric determines a pair of parameter values (λ, μ) since there is one line of each system through such a point.

Going back to the cone (1.1), each point of the quadric is the trace in S_3 of a line of the cone and each line on the quadric is the trace of a plane on the cone. Hence there are two systems of planes on the cone such that two planes of the same system have no point in common except the vertex and such that each plane of one system intersects each plane of the other system in a line. The planes of one system can be designated by the values λ and those of the other system by μ and the pair of values (λ, μ) will determine a line of the cone.

In the case which we are studying, when the equation of the cone is reducible to (1.2), the quadric in S_3 is real but the straight lines on it are imaginary. The quadric is projectively equivalent to an ordinary sphere. Thus the structure of the system of light lines through the vertex of (1.1) is like that of the points on a sphere.

The properties which we have been inferring from theorems of geometry can also be obtained by an elementary algebraic argument. Suppose we set

$$\frac{X^1 + iX^2}{\sqrt{2}} = X^{14} \qquad \frac{X^3 + X^4}{\sqrt{2}} = X^{13}$$
$$\frac{X^1 - iX^2}{\sqrt{2}} = X^{23} \qquad \frac{X^3 - X^4}{\sqrt{2}} = -X^{24}. \tag{2.1}$$

Then

$$2\begin{vmatrix} X^{13} & X^{14} \\ X^{23} & X^{24} \end{vmatrix} = -(X^1)^2 - (X^2)^2 - (X^3)^2 + (X^4)^2. \tag{2.2}$$

If this determinant is to vanish there must exist variables $\psi^1, \psi^2, \psi^3, \psi^4$, such that

$$X^{13} = \psi^1\psi^3 \qquad X^{14} = \psi^1\psi^4$$
$$X^{23} = \psi^2\psi^3 \qquad X^{24} = \psi^2\psi^4. \tag{2.3}$$

Solving the equations (2.1) we obtain

$$
\begin{aligned}
X^1 &= \frac{X^{14} + X^{23}}{\sqrt{2}} = \frac{\psi^1\psi^4 + \psi^2\psi^3}{\sqrt{2}} \\
X^2 &= \frac{X^{14} - X^{23}}{\sqrt{-2}} = \frac{\psi^1\psi^4 - \psi^2\psi^3}{\sqrt{-2}} \\
X^3 &= \frac{X^{13} - X^{24}}{\sqrt{2}} = \frac{\psi^1\psi^3 - \psi^2\psi^4}{\sqrt{2}} \\
X^4 &= \frac{X^{23} + X^{24}}{\sqrt{2}} = \frac{\psi^1\psi^3 + \psi^2\psi^4}{\sqrt{2}}
\end{aligned}
\tag{2.4}
$$

as a parameter representation of the points on the cone (1.1).

If ψ^3 and ψ^4 are held fixed and ψ^1 and ψ^2 regarded as variable parameters these equations say that the point (X^1, X^2, X^3, X^4) describes the plane through the vertex and the points $(\psi^4, -i\psi^4, \psi^3, \psi^3)$ and $(\psi^3, i\psi^3, -\psi^4, \psi^4)$. If ψ^3 and ψ^4 are multiplied through by the same factor, this plane is unaltered. If, however, the ratio ψ^3/ψ^4 is changed the plane is changed and there is one such plane for each value of this ratio. Moreover it is easy to infer from the equations (2.4) that no two of the planes so obtained have any part in common except (0, 0, 0, 0).

In like manner we see that there is another system of planes on the cone, one for each value of ψ^1/ψ^2. Two planes, one from the first system determined by (ψ^3, ψ^4), and one from the second determined by (ψ^1, ψ^2) have a line in common. This line is given by (2.4) with the understanding that the coördinates ψ^1, ψ^2 and ψ^3, ψ^4 may be multiplied through by an arbitrary factor. Thus (2.4) is a parameter representation of the points on the light-cone.

From the geometry of the quadric we know that any linear transformation of the quadric into itself brings about a projective transformation of the generating lines. With this suggestion we subject ψ^1 and ψ^2 to a linear transformation,

$$\bar{\psi}^A = T^A_B\psi^B. \qquad (A, B = 1, 2). \tag{2.5}$$

Substituting this in (2.4) we see that the X's undergo a linear transformation whose coefficients can easily be calculated. The determinant in (2.2) is evidently multiplied by the square of the determinant

$$T = |T^A_B| \tag{2.6}$$

when the transformation (2.5) is carried out. Hence the transformations (2.5) induce transformations of the light-cone into itself. Similarly for linear transformations of ψ^3 and ψ^4.

Instead of regarding (2.5) as a geometric transformation which permutes one system of generating planes of the light cone among themselves, we may regard it as a transformation of the parameters ψ^1, ψ^2 into new parameters, i.e., as a change of the reference system for the light-planes. Because of the physical applications we will refer to ψ^1, ψ^2 (and also ψ^3, ψ^4) as *spin coördinates*. Thus (2.5) can be regarded as a transformation of spin coördinates which brings about a change of the spin-frame of reference.

We shall use the conventions of tensor algebra to indicate the well-known transformation laws that depend on (2.5), for example, the covariant transformation

$$\bar{\psi}_A = t^B_A \psi_B \qquad (A, B = 1, 2) \tag{2.7}$$

where

$$t^A_B T^B_C = \delta^A_C. \tag{2.8}$$

Let us define ϵ_{AB} as having the components

$$\epsilon_{11} = \epsilon_{22} = 0, \quad \epsilon_{12} = -\epsilon_{21} = 1.$$

Then, evidently, the components ϵ_{AB} are the same in all spin-frames if we assume the law of transformation of a relative covariant tensor of weight -1

$$\epsilon_{CD} = \frac{1}{t} \epsilon_{AB} t^A_C t^B_D.$$

The weight is the exponent of the determinant,

$$t = |t^A_B|. \tag{2.9}$$

In like manner we may define

$$\epsilon^{11} = \epsilon^{22} = 0, \quad \epsilon^{12} = -\epsilon^{21} = 1$$

and find

$$\epsilon^{CD} = t\epsilon^{AB} T^C_A T^D_B$$

the transformation law of a relative contravariant tensor of weight $+1$.

We shall use the two ϵ's to raise and lower indices as follows

$$\epsilon_{BA} \psi^B = \psi_A \tag{2.10}$$

or

$$\psi_1 = -\psi^2, \quad \psi_2 = \psi^1$$

$$\epsilon^{AB} \psi_B = \psi^A. \tag{2.11}$$

Notice that in lowering indices we sum with respect to the first index of

ϵ_{AB} while in raising indices we sum with respect to the second index of ϵ^{AB}. This corresponds to the identities

$$-\epsilon_{AB}\epsilon^{CA} = \epsilon_{AB}\epsilon^{AC} = \delta^{C}_{B}. \tag{2.12}$$

It must be remembered that raising indices increases the weight by $+1$ and lowering them decreases it by -1. Thus if we assume ψ^A to be of weight $+^1/_2$ then ψ_A must be of weight $-^1/_2$.

We actually do assume that the ψ^A which enter the formulas (2.4) for the parameterization of the light-cone are of weight $+^1/_2$ because this means that a change of spin-frame causes the transformation

$$\bar{\psi}^A = t^{1/2}T^A_B\psi^B \tag{2.13}$$

which is of determinant 1 and therefore does not change the value of the left-hand member of (2.2).

3. The equations (2.1) can be written in abbreviated form as

$$X^{AP} = g^{AP}_{a}X^{a} \tag{3.1}$$

in which A takes on the values 1 and 2, P the values 3 and 4 and a the values 1, 2, 3, 4. Throughout this paper we shall use the early capital letters, previous to P, as indices to denote the values 1 and 2, and the later capital letters, from P onward, as indices to denote 3 and 4.

The coefficients of (3.1) for the four values of a constitute four 2-row Hermitian matrices

$$g^{\cdot\cdot}_{1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ g^{\cdot\cdot}_{2} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},\ g^{\cdot\cdot}_{3} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ g^{\cdot\cdot}_{4} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{3.2}$$

The equations (2.4) which are inverse to (3.1) may be written,

$$X^{a} = g^{a}_{AP}X^{AP} \tag{3.3}$$

where

$$g^{1}_{\cdot\cdot} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ g^{2}_{\cdot\cdot} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ g^{3}_{\cdot\cdot} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ g^{4}_{\cdot\cdot} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Since (3.1) and (3.3) are inverse, we have

$$g^{AP}_{a}\, g^{b}_{AP} = \delta^{b}_{a} \tag{3.4}$$

and

$$g^{AP}_{a}\, g^{a}_{BQ} = \delta^{A}_{B}\delta^{P}_{Q}. \tag{3.5}$$

Let us now apply the conventions agreed upon in the last section to raising and lowering the indices. Thus

$$g^{P}_{Aa.} = \epsilon_{BA}g^{BP}_{a}\,.$$

This gives four matrices

$$g_{\cdot 1}^{\cdot} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_{\cdot 2}^{\cdot} = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad g_{\cdot 3}^{\cdot} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_{\cdot 4}^{\cdot} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

If we calculate the other possible arrangements of indices we find that the operation of raising and lowering indices as defined in the sections above is consistent with the relations (3.4) and (3.5).

From the equation,

$$g_a^b g_{AP}^{\ a} = g_{AP}^{\ b}$$

it is evident that the g's which appear in this section are all components of the metric tensor, some in the scalar and some in the spin-frames of reference. For example, we have

$$g_{APBQ} = \epsilon_{AB}\epsilon_{PQ}. \tag{3.7}$$

Also we have

$$\delta_B^A g^{ab} = g_P^{Aa} g_B^{Pb} + g_B^{Pa} g_P^{Ab}, \tag{3.8}$$

as may readily be verified.

The components of the metric tensor in the scalar and the spin coördinates are all constants, but this is not true of components, partly in the dx coördinates and partly in the spin coördinates. For example the components in the left-hand member of the equation

$$g_P^{Ai} = g_P^{Aa} g_a^i$$

are not, in general, constants. Nevertheless we have

$$\delta_B^A g^{ij} = g_P^{Ai} g_B^{Pj} + g_P^{Aj} g_B^{Pi}$$

or, in a matrix notation,

$$1 . g^{ij} = g^i g^j + g^j g^i.$$

4. The condition that the coördinates X^1 shall be real gives by (2.1) that

$$X^{13} \text{ and } X^{24} \text{ are real}$$

and

$$X^{14} = X^{23*}$$

where we are using the * to indicate conjugate imaginaries.

From (2.3) it follows that

$$\begin{aligned} \psi^{3*} &= k\psi^1 \\ \psi^{4*} &= k\psi^2 \end{aligned} \tag{4.1}$$

where k is real. The effect of changing the value of k is merely to shift the point X along the same line of the light-cone. Hence no generality is lost by assuming $k = 1$ as we now shall do.

Hence the points X on the light-cone are given by

$$\begin{aligned} X^1 &= \frac{\psi^1\psi^{2*} + \psi^2\psi^{1*}}{\sqrt{2}} \\ X^2 &= \frac{\psi^1\psi^{2*} - \psi^2\psi^{1*}}{\sqrt{-2}} \\ X^3 &= \frac{\psi^1\psi^{1*} - \psi^2\psi^{2*}}{\sqrt{2}} \\ X^4 &= \frac{\psi^1\psi^{1*} + \psi^2\psi^{2*}}{\sqrt{2}}. \end{aligned} \tag{4.2}$$

From the form of (2.4) it is clear that if ψ^1 and ψ^2 are multiplied by a number c and ψ^3 and ψ^4 by $1/c$ the point X represented by (2.4) is unchanged. But if the relation (4.1) is to be undisturbed, we must have

$$cc^* = 1.$$

Hence the point X which is represented by (4.2) is unchanged if we multiply ψ^1 and ψ^2 through by $\exp(i\theta)$ where θ is real. On the other hand a different point on the same generating line of the cone is represented by these equations if we multiply ψ^1 and ψ^2 by a constant whose absolute value is not 1. Each value of the ratio ψ^1/ψ^2 determines a line of the cone.

The parameters ψ like the coördinates X, are unaltered by changes of coördinates in the underlying space-time. But in a given coördinate system x they have a degree of indetermination which is expressed by writing

$$\begin{aligned} \psi^1 &= e^{ix^0}f^1 \\ \psi^2 &= e^{ix^0}f^2 \end{aligned} \tag{4.3}$$

where x^0 is an arbitrary real parameter.

If f^1 and f^2 are taken to be complex functions of the real variables x^1, $\ldots$, x^4 these equations determine a unique real point on the light-cone at each point x. This point is the same for all values of x^0. If we make a transformation

$$\bar{x}^0 = x^0 + \log \rho(x^1, x^2, x^3, x^4) \tag{4.4}$$

where $\log \rho$ is real then f^1 and f^2 are multiplied by $1/\rho$ but the same point on the same light-cone is represented by (4.3). Hence a pair of projective scalars[6] ψ^1 and ψ^2 determine a unique point of the light-cone at each point of space-time.

We may, indeed, make the more general transformation

$$\bar{x}^0 = x^0 + \int \lambda_j dx^j \tag{4.5}$$

instead of (4.4) which changes (4.3) into

$$\psi^A = f^A \exp. (i\bar{x}^0 - i\int \lambda_j dx^j) \tag{4.6}$$

and introduces a parametrization of the light-cone which depends on the path of integration, without changing the point on the light-cone represented by (4.2).

5. Let us now require of our transformations of spin-frame that the reality condition of §4 shall be undisturbed. This means that T^P_Q must be related to T^A_B by the equation

$$T^{A+2}_{B+2} = T^{A*}_B. \tag{5.1}$$

Hence the determinant of the transformation of ψ^3 and ψ^4 is T^*.

As indicated at the end of §2 we subject the spin parameters to the laws of transformation

$$\bar{\psi}^A = t^{1/2}T^A_B\psi^B \tag{5.2}$$

$$\bar{\psi}^P = t^{*1/2}T^P_Q\psi^Q \tag{5.3}$$

and, consistently with this,

$$\bar{X}^{AP} = (t\, t^*)^{1/2}X^{BQ}T^A_BT^P_Q, \tag{5.4}$$

which leaves the quadratic form

$$\begin{vmatrix} X^{13} & X^{14} \\ X^{23} & X^{24} \end{vmatrix}$$

unaltered. Hence

$$L^a_b = (t\, t^*)^{1/2}g^{\ a}_{AP}g^{BQ}_b T^A_B T^P_Q \tag{5.5}$$

is a Lorentz transformation.

6. We now have before us a new class of geometrical objects with the following characteristics. Consider two functions,

$$\begin{aligned} \psi^1 &= e^{ix^0}f^1(x^1, x^2, x^3, x^4) \\ \psi^2 &= e^{ix^0}f^2(x^1, x^2, x^3, x^4). \end{aligned} \tag{6.1}$$

If there is fixed a coördinate system, a gauge and a spin-frame, then the equations (6.1) together with (4.2) determine a definite point on each light-cone. Under a transformation of coördinates or of gauge ψ^1 and ψ^2 behave like projective scalars of index i, that is, the transformation is effected by substituting

$$\begin{aligned} x^0 &= \bar{x}^0 - \log \rho(\bar{x}) \\ x^i &= x^i(\bar{x}) \end{aligned} \tag{6.2}$$

in (6.1). We may also allow the non-holonomic gauge transformation (4.5). A change of the spin-frame requires that ψ^1 and ψ^2 be replaced by $\bar{\psi}^1$ and $\bar{\psi}^2$ according to (5.2). The geometrical object having components ψ^1 and ψ^2 subject to all these conditions, we will call a contravariant *spinor* of the first order of weight $^1/_2$. Spinors of other orders and weights are defined by replacing (5.2) by the suitable linear relations after the fashion of tensor algebra.

7. In order to have a theory of covariant differentiation of spinors we have only to introduce the analogue of an affine connection. This is a geometric object whose 20 components

$$\Lambda^A_{B\alpha} \quad (A, B = 1, 2; \ \alpha = 0, 1, 2, 3, 4)$$

are functions of x^1, x^2, x^3, x^4, alone and under changes of spin-frame obey the transformation law

$$\bar{\Lambda}^C_{D\alpha} = \left(\Lambda^A_{B\alpha} t^B_D + \frac{\partial t^A_D}{\partial x^\alpha}\right) T^C_A \tag{7.1}$$

whereas under transformation of coördinates and gauge they behave like components of a projective covariant vector. We shall call this a *spinor connection of the first kind.*

It then follows by the same calculation as in tensor analysis[7] that if ψ^A is a spinor of weight N

$$\frac{\partial\psi^A}{\partial x^\alpha} + \Lambda^A_{B\alpha}\psi^B - N\,\Lambda^B_{B\alpha}\psi^A = \psi^A_{,\alpha} \tag{7.2}$$

are the components of a geometric object which transform like those of a spinor of weight N with respect to the index A and like those of a projective tensor with respect to α. That is to say if we change coördinates, gauge and spin-frame all at once we have

$$\bar{\psi}^A_{,\alpha} = t^{1/2}\psi^B_{,\beta} T^A_B \frac{\partial x^\beta}{\partial \bar{x}^\alpha}. \tag{7.3}$$

We call $\psi^A_{,\alpha}$ the covariant derivative of ψ^A.

Covariant differentiation of spinors of arbitrary order and weight is defined analogously by the same formulas as in tensor analysis.[7] For example, if x^{AP} is a spinor with the law of transformation (5.4)

$$X^{AP}_{,\alpha} = \frac{\partial X^{AP}}{\partial x^\alpha} + \Lambda^A_{B\alpha}X^{BP} + \Lambda^P_{Q\alpha}X^{AQ} - \frac{1}{2}\Lambda^B_{B\alpha}X^{AP} - \frac{1}{2}\Lambda^Q_{Q\alpha}X^{AP}. \tag{7.4}$$

Our reality conditions make it necessary, of course, that

$$\Lambda^{A+2}_{B+2\alpha} = \Lambda^{A*}_{B\alpha}. \tag{7.5}$$

Hence

$$\Lambda^P_{P\alpha} = \Lambda^{A*}_{A\alpha}. \tag{7.6}$$

If $\Lambda^A_{B\alpha}$ are components of a spinor connection of the first kind the functions

$$\Gamma^A_{B\alpha} = \Lambda^A_{B\alpha} - \frac{1}{2}\Lambda^C_{C\alpha}\delta^A_B \tag{7.7}$$

are the components of a geometric object which we shall call a *spinor connection of the second kind.* Its law of transformation under changes of spin-frame is

$$\bar{\Gamma}^A_{B\alpha} = \left(\Gamma^C_{D\alpha}t^D_B + \frac{\partial t^C_B}{\partial x^\alpha}\right) T^A_C - \frac{1}{2}\frac{\partial \log t}{\partial x^\alpha}\delta^A_B \tag{7.8}$$

and it satisfies the invariant condition

$$\Gamma^A_{A\alpha} = 0. \tag{7.9}$$

In case $N = {}^1/_2$, that is in case ψ^A is a spinor of weight ${}^1/_2$ and index i, (7.2) may be written

$$\frac{\partial\psi^A}{\partial x^\alpha} + \Gamma^A_{B\alpha}\psi^B = \psi^A_{,\alpha} \tag{7.10}$$

8. The formula (7.10) for covariant differentiation of a spinor can be used to obtain a displacement formula as follows The equations

$$\psi^A_{,\alpha} - \psi^A_{,0}\varphi_\alpha = 0 \tag{8.1}$$

where φ_α is an arbitrary projective vector such that

$$\varphi_0 = 1, \tag{8.2}$$

are invariant under transformations of coördinates, gauge and spin-frame. Also they are satisfied identically for $\alpha = 0$. Hence the equations

$$(\psi^A,\ i - \psi^A_{,0}\varphi_i)\frac{dx^i}{dt} = 0 \tag{8.3}$$

are invariant under all three types of transformation.

It is sufficient for our present purposes if we limit attention to the case in which[8]

$$\Gamma^A_{B0} = 0. \tag{8.4}$$

Under these conditions the components Γ^A_{Bi} are unaltered by gauge transformations and respond to transformations of coördinates like the components of a covariant affine vector. Substituting (7.10) in (8.3) we now obtain

$$\frac{d\psi^A}{dt} + \Gamma^A_{Bj}\psi^B \frac{dx^j}{dt} = i\varphi_j \frac{dx^j}{dt}\psi^A. \tag{8.5}$$

This is the formula for the displacement of the ψ's which enter in the parameterization (4.3) of the light-cone along a curve

$$x^i = x^i(t).$$

Applying this and the corresponding formula for the conjugate imaginary ψ's to

$$X^{AP} = \psi^A\psi^P$$

we find the following displacement formula,

$$\frac{dX^{AP}}{dt} + \Gamma^A_{Bj}X^{BP}\frac{dx^j}{dt} + \Gamma^P_{Qj}X^{AQ}\frac{dx^j}{dt} = 0, \tag{8.6}$$

which in the scalar coördinates of §1 becomes

$$\frac{dX^a}{dt} + \Gamma^a_{bj}X^b\frac{dx^j}{dt} = 0, \tag{8.7}$$

where

$$\Gamma^a_{bj} = g^a_{AP}\,\Gamma^A_{Bj}g^{BP}_b + g^a_{AP}\Gamma^P_{Qj}g^{AQ}_b. \tag{8.8}$$

These functions satisfy the condition,

$$\Gamma^a_{aj} = 0 \tag{8.9}$$

in consequence of (7.9).

9. The equation (8.7) is the formula for an affine displacement and we can in fact identify this displacement with the displacement by infinitesimal parallelism determined by the gravitational tensor g_{ij}. The latter is given by

$$\frac{dX^i}{dt} + \Gamma^i_{jk}X^j\frac{dx^k}{dt} = 0 \tag{9.1}$$

in the coördinates x, where $X^i = g^i_a X^a$ and Γ^i_{jk} are the Christoffel symbols of the second kind formed from g_{ij}. Under the transformation (1.2) we find

$$\Gamma^a_{bj} = \left(\Gamma^i_{kj}g^a_i - \frac{\partial g^a_k}{\partial x^j}\right)g^k_b \tag{9.2}$$

and also, as is well known, that these functions satisfy the condition (8.9).

The equation (8.8) now yields

$$g^{AP}_a\Gamma^a_{bj}g^b_{BQ} = \Gamma^A_{Bj}\delta^P_Q + \Gamma^P_{Qj}\delta^A_B. \tag{9.3}$$

Setting $A = B$ and $P = Q$ and summing we obtain

$$\Gamma^{A}_{Aj} = -\Gamma^{P}_{Pj} = \lambda_j. \tag{9.4}$$

We obtain

$$\Gamma^{A}_{Bj} - \delta^{A}_{B}\lambda_j = {}^1/_2\Gamma^{a}_{bj}g_a^{AP}g^{b}_{BP} \tag{9.5}$$

by setting $P = Q$ in (9.3) and summing. This equation determines Γ^{A}_{Bj} uniquely if in accordance with (7.9) we assume $\lambda_j = 0$. Similarly we obtain

$$\Gamma^{P}_{Qj} = {}^1/_2\Gamma^{a}_{bj}g_a^{AP}g^{b}_{AQ}. \tag{9.6}$$

Thus a displacement (9.1) determines a pair of conjugate spin displacements

$$\nabla^{PB}_{AC}\psi^C = 0, \quad \nabla^{AQ}_{PR}\psi^R = 0,$$

where

$$\nabla^{PB}_{AC} = g^{Pj}_{A}\left(\delta^{B}_{C}\frac{\partial}{\partial x^j} + \Gamma^{B}_{Cj} - i\varphi_j\delta^{B}_{C}\right)$$

and

$$\nabla^{AQ}_{PR} = g^{Aj}_{P}\left(\delta^{Q}_{R}\frac{\partial}{\partial x^j} + \Gamma^{Q}_{Rj} + i\varphi_j\delta^{Q}_{R}\right).$$

The functions φ_j are completely arbitrary.

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[3] V. Fock, "Geometrisierung der Diracschen Theorie des Elektrons," *Ibid.*, **57**, 261 (1929).

[4] J. A. Schouten, "Dirac Equations in General Relativity," *Jour. Math. and Phys.*, **10**, 239 (1931).

[5] B. L. van der Waerden, "Spinoranalyse," *Göttingen Nachrichten*, 100 (1929).

[6] For the definition of projective scalars and of projective tensors in general, see O. Veblen, "Projektive Relativitätstheorie," *Ergebnisse der Math.*, **2**, Berlin, 1933.

[7] See for example O. Veblen, *Invariants of Quadratic Differential Forms*, Cambridge, pp. 37, 38 (1927).

[8] The introduction of this simplifying assumption at this point was suggested by Mr. J. L. Vanderslice.